Fundamental Approach to Equivalent Systems Analysis

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The use of low-order approximations, or equivalent systems, in evaluating aircraft handling qualities is reviewed. This problem is identified as a special case of the more general problem of model reduction in closed-loop systems. Critical characteristics that must be reflected in the approximation, as well as factors influencing the likelihood of obtaining the appropriate low-order approximation, are discussed. An alternate procedure is offered that allows for the evaluation of more complex, multiloop piloted tasks and avoids a numerical search procedure.

Introduction

A CRITICAL review is presented, and an alternate extended procedure is offered, for the equivalent system technique[1] for evaluating aircraft handling qualities. In the equivalent systems approach, a numerical search algorithm has been employed to find the reduced-order model, of "classical" aircraft form, such that the frequency response of the higher-order system (aircraft) is well approximated over a specified frequency range. Questions have been raised, however especially when a good approximation is not obtained with the method, and are related to the following[2]: 1) the nonuniqueness of solutions, 2) the interpretation of the matching cost, 3) the "goodness of fit" required, 4) the uncertainty as to whether to fix or free parameters in the lower order transfer functions, 5) the appropriate treatment of the multi-input/multi-output case, and 6) the concept of effective system dynamic order. The origins of some of these problems may be generated by the reduction procedure, while others may result from not considering certain aspects of closed-loop system analysis or from the inability to accurately define the task. And still others may arise because each system possesses an effective order, often times higher than that desired.

Because of some of these fundamental difficulties, the reduced-order modeling objective of approximating the aircraft's frequency response is re-examined, and when and how to match multiple frequency responses will be reviewed. An alternate state-space model-reduction approach will be proposed. The original transfer function (matrix) $G(s)$ of dynamic order n is to be reduced, via a state-space transformation T. The construction of T involves no numerical search algorithm. Furthermore, the resulting model G_r is unique for the selected dynamic order r. The least effective dynamic order is determined a priori by evaluating a set of frequency-domain matching error bounds. These error bounds, furthermore, are applicable to each i-j element of $[G(s) - G_r(s)]_{s=j\omega}$ over all ω. These error bounds may naturally be interpreted on a Bode plot. Finally, the approach is applicable to multi-input/multi-output systems.

Factors Critical to Handling Qualities Evaluation

The governing factor in the construction of any reduced-order model is its intended application. The intended application of the equivalent lower-order system (LOS) is to analyze and evaluate a higher-order system's (HOS) (i.e., aircraft) handling qualities. With this objective in mind, three fundamental aspects of handling qualities pointed to by Harper[3] should be noted. First, handling is a dynamic problem. Second, the system dynamics being considered during the handling qualities evaluation include the pilot and the vehicle acting together. Consequently, the LOS model must be constructed such that the closed-loop structure is kept in mind. Third, the dynamics of the pilot/vehicle system differ as the pilot performs different tasks. Therefore, with respect to model reduction in handling qualities analysis, the task evaluated dictates which responses should be approximated by the LOS.

Nothing underscores these concepts better than the work that appeared in the late 1950's and early 1960's[4] promoting a systems analysis approach to pilot/vehicle closed-loop control. Noteworthy was the work of Neal and Smith,[5] which took these ideas and produced a predictive handling qualities analysis tool for a particular task. The implications of the Neal-Smith analysis technique to the equivalent-system approach are as follows. Since the pilot's rating is an assessment of the aircraft's handling qualities in this task, and because the equivalent LOS hypothetically possesses the handling quality deficiencies of the HOS vehicle, the equivalent LOS should elicit the same pilot compensation, produce the same performance, and yield the same results as the HOS vehicle when the loop is closed. To do this, the LOS model must at least approximate the original's frequency response in the region of open-loop pilot-plus-vehicle gain crossover.[6] It is for this reason that the frequency response is the key input/output characteristic to match. Yet, if the loop gain is high at frequencies below crossover, and is attenuated well at frequencies above crossover, a close match is required only in the crossover region.

To illustrate this point, consider the block diagram shown in Fig. 1. The closed-loop system consists of the pilot's compensation P_θ acting upon the attitude tracking error θ_e and a mathematical representation of the aircraft, $\theta/F_s(s)$. θ_c is the commanded attitude. Depending on whether the switch is open or closed, the aircraft is represented by an LOS approximation or by its higher-order transfer function. A Neal-Smith analysis is to be performed on the HOS via the LOS model.

The HOS system to be considered is configuration 1A from the Neal-Smith study.[5]

$$\left(\frac{\theta}{F_s}\right)_{\text{HOS}} = \frac{0.53(1.25)(49.06)}{(0)(2.0)(1.518)[0.69,2.2][0.75,63]} \frac{\text{deg}}{\text{lb}} \tag{1}$$

The LOS approximation was obtained using the methodology to be described in this paper.

$$\left(\frac{\theta}{F_s}\right)_{\text{LOS}} = \frac{10.73(0.444)e^{-.01s}}{(0)[0.62,2.47]} \frac{\text{deg}}{\text{lb}} \tag{2}$$

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As in Neal-Smith, a pilot compensation model was then constructed to control $(\theta/F_s)_{\mathrm{LOS}}$.

$$P_\theta(s) = 0.078(s+8.6)e^{-.3s}\ (\mathrm{lb/deg}) \tag{3}$$

which can be shown to be the same pilot compensation needed to control $(\theta/F_s)_{\mathrm{HOS}}$. All three frequency responses are shown in Fig. 2.

From plots of the open- and closed-loop responses, for example, it may be shown that this same pilot compensation produced the same low frequency performance (droop = −3 dB) and the same resonant peak in both systems. Consequently, a Neal-Smith analysis on either representation of the vehicle dynamics would yield the same prediction of pilot rating thereby demonstrating the critical significance of matching the frequency response in the crossover region.

For multiloop tasks, the generalized "crossover" region[7] has the same significance. Assuming $P(j\omega)$ represents the pilot's multiloop compensation, this generalized crossover region is determined by the minimum and maximum singular value of the complex matrix $GP(j\omega)$, i.e.,

$$\begin{aligned}\underline{\sigma}[GP(j\omega_a)] &\approx 1\\ \bar{\sigma}[GP(j\omega_b)] &\approx 1\end{aligned} \tag{4}$$

where $G(s)$ is the task dependent multiloop aircraft transfer function matrix. Define (ω_1,ω_2) so that $\omega_1 < \omega_a < \omega_b < \omega_2$. Analogous to the single loop version, the multivariable crossover also determines relative stability (robustness), bandwidth, and the closed-loop system's dominant modes, implying that the frequency response of every element in the transfer function matrix relevant to the task being evaluated should be well approximated in this region.[6]

Model Reduction Criteria

The problem under consideration may then be stated as follows. Given a $p \times m$ transfer function matrix $G(s)$, find a reduced-order model $G_r(s)$ (also $p \times m$) such that $G(s)$ approximates $G_r(s)$ in some meaningful way. In multi-input/multi-output closed-loop systems, which include pilot/vehicle systems, "meaningful" implies that $G_r(j\omega)$ must approximate $G(j\omega)$ element by element over the multivariable crossover frequency range. Designate this range by (ω_1,ω_2), i.e., $\omega_1 < \omega < \omega_2$.

To be specific, the frequency response error is defined by

$$E(j\omega) = [E_{ij}(j\omega)] \triangleq G(j\omega) - G_r(j\omega) \tag{5}$$

Each *i-j* element in $E(j\omega)$ describes the frequency response error associated with the corresponding element in $G(j\omega)$. For any frequency ω $(\omega_1 < \omega < \omega_2)$, $E_{ij}(j\omega)$ is the difference between two complex numbers. If

$$\begin{aligned}G_{ij}(j\omega) &= a_{ij} + jb_{ij}\\ G_{r_{ij}}(j\omega) &= \hat{a}_{ij} + j\hat{b}_{ij}\end{aligned} \tag{6}$$

then

$$E_{ij}(j\omega) = (a_{ij} - \hat{a}_{ij}) + j(b_{ij} - \hat{b}_{ij})$$

for $G_{r_{ij}}(j\omega)$ to approximate $G_{ij}(j\omega)$, then

$$|E_{ij}(j\omega)| = [(a_{ij} - \hat{a}_{ij})^2 + (b_{ij} - \hat{b}_{ij})^2]^{\frac{1}{2}} \tag{7}$$

must be small compared to $|G_{ij}(\omega)|$.

A matrix norm, defined by the maximum singular value of the matrix $E(j\omega)$, provides a measure of "smallness" for $E(j\omega)$, and also bounds $|E_{ij}(j\omega)|$. Recall the maximum singular value of E, which is defined as

$$\bar{\sigma}(E) = \bar{\lambda}^{\frac{1}{2}}(EE^*) \tag{8}$$

satisfies

$$\bar{\sigma}(E) = \max_{\|v\|=1} (v^*EE^*v)^{\frac{1}{2}} \tag{9}$$

where $\bar{\lambda}(\cdot)$ is the maximum eigenvalue of $(\cdot)$, and $\|v\| = (v^*v)^{\frac{1}{2}}$. The implication of Eq. (9) is easily demonstrated on a 2×2 error transfer function matrix

$$E(s)|_{s=j\omega} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

with $v^* = [1\ 0]$, i.e. $\|v\| = 1$,

$$(v^*EE^*v) = |E_{11}|^2 + |E_{12}|^2$$

and

$$\bar{\sigma}(E) \geq (|E_{11}|^2 + |E_{12}|^2)^{\frac{1}{2}} \tag{10}$$

Likewise, with $v^* = [0\ 1]$ it can be shown

$$\bar{\sigma}(E) \geq (|E_{21}|^2 + |E_{22}|^2)^{\frac{1}{2}} \tag{11}$$

A similar result holds for the magnitude of the column vectors of E since $\bar{\sigma}(E) = \bar{\sigma}(E^*)$. From Eqs. (10) and (11), $\bar{\sigma}(E)$ bounds the magnitude of each element in $E(j\omega)$.

The largest value of $\bar{\sigma}[E(j\omega)]$ over all frequencies $(0 \leq \omega < \infty)$ defines the "∞ norm"

$$\|E(j\omega)\|_\infty = \sup_\omega \bar{\sigma}[E(j\omega)] \tag{12}$$

If for some ω $\|E(j\omega)\|_\infty \ll |G_{ij}(j\omega)|$, then $G_{r_{ij}}(j\omega)$ closely approximates $G_{ij}(j\omega)$. Interest in this norm stems from the fact

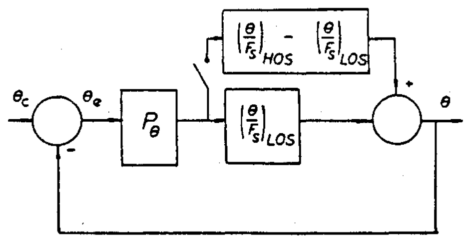

Fig. 1 Closed-loop system schematic.

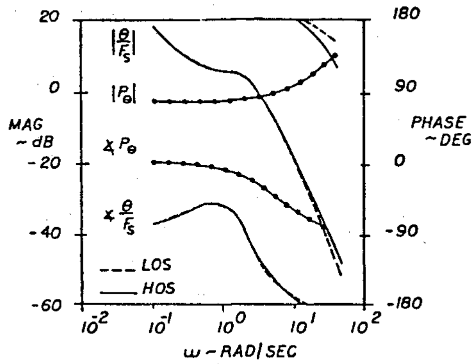

Fig. 2 Closed-loop example: pilot and aircraft frequency responses.

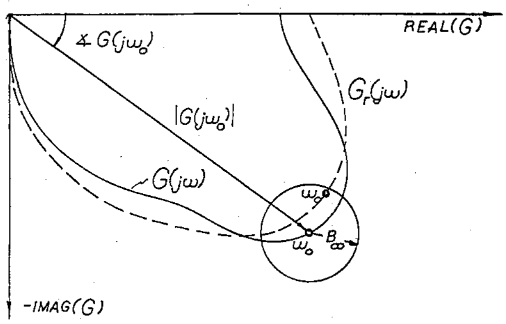

Fig. 3 Geometric interpretation B_∞.

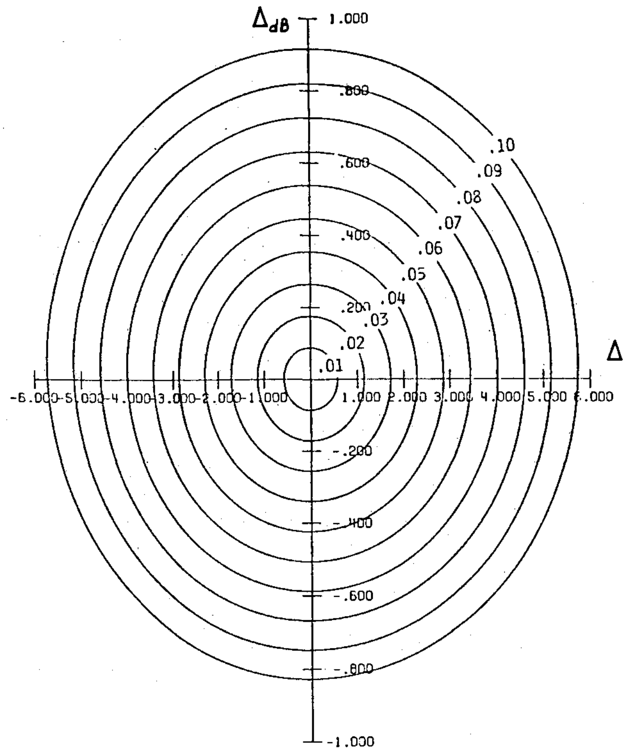

Fig. 4 Constant α contours.

that the model reduction procedure to be presented exploits another reduction technique, i.e., Moore's internally balanced approach.[8] Enns[9] has shown that a model obtained from Moore's technique satisfies

$$\|E(j\omega)\|_\infty \leq B_\infty \tag{13}$$

where the value of B_∞ may be determined before the reduction is performed. The procedure to be presented also has such a bound.

A graphical interpretation of Eq. (13) is provided in Fig. 3, using a polar plot of the *i-j* element $G(j\omega), G_{ij}(j\omega)$. For any ω, Eq. (13) implies that $G_{r_{ij}}(j\omega)$ must lie within a circle of radius B_∞ centered about $G_{ij}(j\omega)$. To translate this error bound to the Bode diagram, the interior of this circle must be translated into a set of Bode errors $[\Delta_{dB}, \Delta_o]$. From Fig. 3, given B_∞, the set of $[\Delta_{dB}, \Delta_o]$ is dependent on the magnitude of $G_{ij}(j\omega)$. To be specific, it is dependent on the ratio $B_\infty/|G_{ij}(j\omega)|$.

If α is the ratio generically expressed as

$$\alpha = (B_\infty/M_{II}) \tag{14}$$

where M_{II} is some positive scalar (not in dB) greater than B_∞, then the set of possible Bode errors $\{\Delta_{dB}, \Delta_o\}$ for those frequencies ω for which

$$|G_{ij}(j\omega)| \geq M_{II} \tag{15}$$

is enclosed by the boundary $(\Delta_{dB_B}, \Delta_{o_B})$ where

$$\Delta_{dB_B} = -10\log(1+\alpha^2+2\alpha\cos\phi)$$
$$\Delta_{o_B} = -\tan^{-1}\left(\frac{\alpha\sin\phi}{1+\alpha\cos\phi}\right) \tag{16}$$

$-\pi < \phi < \pi$. Contours of constant α are presented in Fig. 4.

Now consider the present equivalent systems approach,[1] for which a good approximation is considered to be one with a mismatch (cost) of less than or equal to 10, ($J \leq 10$), where

$$J = \sum_{i=1}^{20} \{[\Delta_{dB}(\omega_i)]^2 + 0.01745[\Delta_o(\omega_i)]^2\} \tag{17}$$

Here, $\Delta_{dB}(\omega_i)$ and $\Delta_o(\omega_i)$ are the difference in gain and phase, respectively, betweeen the HOS and LOS frequency responses evaluated at ω_i. If it is assumed that

$$\{[\Delta_{dB}(\omega_i)]^2 + 0.01745[\Delta_o(\omega_i)]^2\}$$
$$\approx \{\Delta[_{dB}(\omega_{i+1})]^2 + 0.01745[\Delta_o(\omega_{i+1})]^2\} \tag{18}$$

for each $i = 1, 19$, then if $J \leq 10$, these Bode errors must lie within the ellipse

$$\frac{(\Delta_{dB})^2}{(0.7071\text{ dB})^2} + \frac{(\Delta_o)^2}{(5.347°)^2} \leq 1 \tag{19}$$

which roughly corresponds to $\alpha \leq 0.1$ in Fig. 4.

Unfortunately, minimizing the sum in J does not ensure Eq. (18), and thus $J \leq 10$ does not imply that each set of errors over (ω_1, ω_2) lies within the bounds defined in Eq. (19). Moreover, the problem is compounded when more than one response is modeled, since the J being minimized is an average cost over all being modeled. In contrast, $\alpha < 0.1$ over each element of $G(j\omega)$ does imply that a certain Bode error bound is achieved for all $\omega_1 < \omega < \omega_2$. Such a bound is critical if a certain stability margin, level of performance, and closed-loop resonance are to be preserved, and the method to be proposed satisfies such a bound.

Modal Decomposition

Because modal decomposition will be used, the implications in terms of the system's frequency response will be noted. The system's transfer function $G(s)$ (assumed scalar initially) describing the input/output behavior of the linear system is

$$G(s) = \frac{Q(s)}{P(s)} = \frac{\gamma\prod^{m}(s-z_i)}{\prod^{n}(s-p_i)} = \sum^{n}\frac{R_i}{(s-p_i)} \tag{20}$$

where R_i denotes the (impulse) residue associated with pole p_i. It is these parameters, the poles, zeroes, and residues, that influence the frequency response and the systems effective order, which shall be addressed.

The magnitude and phase on the Bode diagram of $G(j\omega)$ are related to the directed line segments shown in Fig. 5 as

$$|G(j\omega)|_{dB} = 20\left[\log|\gamma| + \sum^{m}\log|j\omega - z_i| - \sum^{n}\log|j\omega - p_i|\right] \tag{21}$$

$$\angle[G(j\omega)] = \angle\gamma + \sum^{m}\angle(j\omega - z_i) - \sum^{n}\angle(j\omega - p_i) \tag{22}$$

where the angles are justified as in the figure. As ω moves along the imaginary axis, these directed segments, of course, rotate

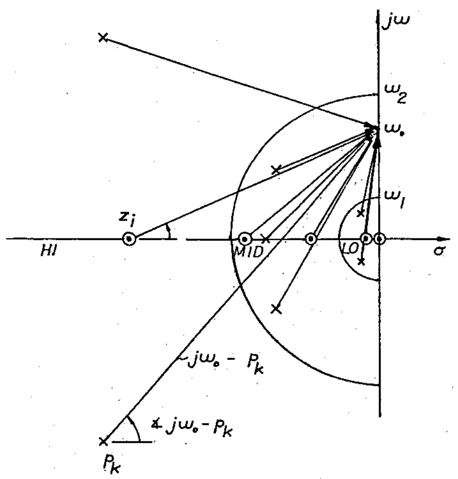

Fig. 5 Pole/zero definition of frequency response.

and change magnitude. Of particular interest to this study is constraining ω to lie within (ω_1,ω_2).

Two concentric circles of radii ω_1 and ω_2 have been drawn about the origin. These circles separate the s plane into a low-, mid-, and high-frequency region, and can be used to separate the pole/zero constellation into three sets. The HOS transfer function may be rewritten as

$$G(s) = \gamma\left(\frac{\prod^{m_{\mathrm{LO}}}(s-z_i)}{\prod^{n_{\mathrm{LO}}}(s-p_i)}\right)\left(\frac{\prod^{m_{\mathrm{MID}}}(s-z_i)}{\prod^{n_{\mathrm{MID}}}(s-p_i)}\right)\left(\frac{\prod^{m_{\mathrm{HI}}}(s-z_i)}{\prod^{n_{\mathrm{HI}}}(s-p_i)}\right) \tag{23}$$

Poles and impulse residues may likewise be grouped according to the pole's location in the s-plane, or consider

$$G(s) = G_{\mathrm{LO}}(s) + G_{\mathrm{MID}}(s) + G_{\mathrm{HI}}(s) \tag{24}$$

$$G_k(s) = \left(\sum_i^{n_k}\frac{R_i}{s-p_i}\right)_k, \qquad k = \mathrm{lo,mid,hi} \tag{25}$$

Modal participation in the system's frequency response at ω is defined by the partial fraction evaluated at ω. Modal dominance at ω is then quantified by the relative magnitudes of these partial fractions.

The residue corresponding to the ith pole

$$R_i = [(s-p_i)G(s)]_{s=p_i} = \frac{\gamma\prod_{j=1}^{m}(p_i-z_j)}{\prod_{k\neq i}^{n}(p_i-p_k)} \tag{26}$$

can also be interpreted geometrically in the complex plane. To be noted, near pole/zero cancellations translate into small residues for the corresponding modes. Also note, for example, that if modal truncation is used in the model reduction to eliminate the low-frequency subsystem G_{lo}, the turncation error introduced is

$$E_t(s) = G_{\mathrm{LO}}(s) = \left(\sum_i^{n_{\mathrm{LO}}}\frac{R_i}{s-p_i}\right)_{\mathrm{LO}} \tag{27}$$

If, after such a truncation, the resulting model is then residualized, creating

$$\tilde{G}(s) \triangleq G_{\mathrm{MID}}(s) + [G_{\mathrm{HI}}(s)]_{s=0} \tag{28}$$

the error due to this residualization step may be expressed as

$$E_r(s) = \left(-\sum_i^{n_{\mathrm{HI}}}\left(\frac{R_i}{p_i}\right)\frac{s}{(s-p_i)}\right)_{\mathrm{HI}} \tag{29}$$

From these expressions for the errors, it is noted that truncation will lead to a reduced-order model that well approximates the frequency response at frequencies well above the modes truncated and/or when the truncated modal residue is small. Residualization will lead to lower-order systems that well approximate the original system's frequency response at frequencies well below the frequencies of the modes residualized, and/or when the step-residues (R_i/p_i) of those modes are small. Finally, note that Eqs. (24), (25), (27–29) have multivariable generalizations in that $G(s)$ and residues R_i may be matrices.

Effective System Order

Now consider the system $G(s)$, in general multivariable, where

$$G(s) = C(sI-A)^{-1}B \tag{30}$$

The poles and residue matrices of $G(s)$ are most easily identified by transforming the system into modal form

$$\dot{\bar{x}} = \Lambda\bar{x} \times \bar{B}u \qquad \dim(u) = m \tag{31}$$

$$y = \bar{C}\bar{x} \qquad \dim(y) = p$$

and now

$$G(s) = \bar{C}(sI-\Lambda)^{-1}\bar{B} \tag{32}$$

where $\bar{B} = M^{-1}B$, $\bar{C} = CM$, Λ is the diagonal matrix of eigenvalues of A, and M is the modal matrix of A. The residues are constructed from the rows of $\bar{B},\bar{b}_i^T$, and the columns of $\bar{C},\bar{c}_i$, in the dyadic product of Eq. (33).

$$G(s) = \sum^{n}\frac{\bar{c}_i\bar{b}_i^T}{s-p_i} \tag{33}$$

The modal participation of p_i is now determined by the size and rank of the product $\bar{c}_i\bar{b}_i^T$ and the closeness of $s = p_i$ to the imaginary axis.

Zero residue matrices can result from modes that are either uncontrollable $(\bar{b}_i^T = \bar{0})$, or unobservable $(\bar{c}_i = \bar{0})$, or both. State-space representations which have no zero residue matrices are minimal realizations, since they have the minimal number of modes required to determine the input/output behavior of the system.

The question of whether a mode is controllable or observable is clearly defined. Less clear is quantifying how controllable or how observable a mode is, or whether it even makes sense to quantify, since the product $\bar{c}_i\bar{b}_i^T$ defines the residues. Another set of parameters can be used to gage the system's effective order. These are the system's Hankel singular values and are the square roots of the eigenvalues of the matrix product XS[8]

$$h_i = \lambda_i^{\frac{1}{2}}(XS), \qquad h_i \geq h_{i+1} \tag{34}$$

where X is the controllability grammian and S the observability grammian.

Assuming the eigenvalues of A have negative real parts, the controllability grammian X is the unique solution of

$$XA^T + AX + BB^T = 0 \tag{35}$$

(The range of X is equal to the subspace spanned by the columns of M pertaining to poles with nonzero $\bar{b}_i^T$.) The observability grammian S is the unique solution

$$SA + A^TS + C^TC = 0 \tag{36}$$

(The null space of S is equal to subspace spanned by the columns of M pertaining to poles with zero $\bar{c}_i$.) Although both X and S are dependent on the particular state-space representation (A,B,C), the Hankel singular values, like residues and the transfer function, are not.

The rank of XS equals the number of modes that are both controllable and observable, i.e., the minimal order.[10] It has been shown[9] that this product is always diagonalizable and its eigenvalues are real and non-negative. Consequently, the rank of XS equals n minus the number of zero Hankel singular values. Because of factors that determine rank in this case, relatively small Hankel singular values would indicate the presence of modes with relatively small modal participation (near pole/zero cancellations, small residues). The ratio h_r/h_{r+1}, where h_i are ordered such that $h_i \geq h_{i+1}$, could be used to infer effective order.

To illustrate, consider the following linear, time-invariant system

$$\dot{x} = Ax + Bu, \quad y = Cx$$

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ \varepsilon \\ \varepsilon \end{bmatrix}, \quad C = [1\ \varepsilon\ 1\ \varepsilon] \tag{37}$$

with transfer function

$$G(s) = C(I_n s - A)^{-1}B = \frac{1}{s+1} + \frac{\varepsilon}{s+2} + \frac{\varepsilon}{s+3} + \frac{\varepsilon^2}{s+4} \tag{38}$$

For $\varepsilon = 0$, the system contains a controllable and observable mode, a controllable and unobservable mode, an uncontrollable and observable mode, and an uncontrollable and unobservable mode. The transfer function equals $G_1(s) = 1/(s+1)$ and the Hankel singular values are (0.5,0.0,0.0,0.0). The minimal order is one.

For $\varepsilon \neq 0$, all modes are controllable and observable and the minimal order is four. The transfer function for $\varepsilon = 0.1$, for example, is

$$G_2(s) = \frac{1.21(s+1.90)(s+2.85)(s+3.97)}{(s+1)(s+2)(s+3)(s+4)} \tag{39}$$

and the Hankel singular values are (0.536, 0.685×10^{-2}, 0.865×10^{-4}, 0.744×10^{-6}). Here, the large separation between h_1 and h_2, $h_1/h_2 \gg 1$, indicates that the effective order of $G_2(s)$ is one, which can be verified in Eq. (39) by the near pole-zero cancellations.

The Hankel singular values, the controllability grammian, and the observability grammian play an integral part in the internally balanced approach to model reduction, shown in Table 1. In the reduction, the system's state-space is decomposed via a spectral decomposition of the matrix product $XS = T\Sigma^2 T^{-1}$ (i.e., the columns of T are the eigenvectors of XS) into two components, one more controllable/observable than the other. The more controllable/observable component defines the state-space of the system's low-order approximation. Key to the internally balanced approach's success is that controllability and observability are considered together, a must in light of the residue's definition.

Retaining the r more controllable/observable components in the state-space approach tends to imply that the r components with the largest modal participation, (i.e., components producing the largest frequency response magnitudes) will be the ones approximated. When $\Sigma_{n-r} = 0$, the procedure obtains a minimal realization in much the same way as by the General Decomposition Theorem.[11] In the example, the first-order internally balanced approximation of $G_1(s)$ is

$$\hat{G}_1(s) = \frac{1}{s+1} = G_1(s) \tag{40}$$

Table 1 Internally balanced approach (scaled)

Given: HOS state-space description A,B,C

input scaling $Q_i = \text{diag}(q_{ik})$ $(1 \leq k \leq m)$

output scaling $Q_o = \text{diag}(q_{ok})$ $(1 \leq k \leq p)$

Find: rth order LOS

STEP 1. Solve for X and S

$$AX + XA^T + BQ_i^2 B^T = 0$$

$$A^T S + SA + C^T Q_o^2 C = 0$$

STEP 2. Find T and Λ where $XS = T\Lambda T^{-1}$, $T = [T_r, T_{n-r}]$, $T^{-T} = [U_r, U_{n-r}]$ and

$$\Lambda = \begin{bmatrix} \Sigma_r^2 & 0 \\ 0 & \Sigma_{n-r}^2 \end{bmatrix}$$

where $\Sigma_r = \text{diag}(h_i)$, $i = 1, r$; $\Sigma_{n-r} = \text{diag}(h_i)$, $i = r+1, n$ and

$$h_1 \geq h_2 \geq \cdots \geq h_r > h_{r+1} \geq \cdots \geq h_n \geq 0$$

STEP 3. LOS is defined by

$$A_r = U_r^T A T_r, \quad B_r = U_r^T B, \quad C_r = C T_r$$

When $\Sigma_{n-r} \neq 0$, but $\Sigma_r \gg \Sigma_{n-r}$, the poles and residues of the LOS system will not be a subset of the original, but will change to compensate for those components discarded. The first-order internally balanced approximation of $G_2(s)$ is

$$G_2(s) = \frac{1.17}{s+1.09} \tag{41}$$

As a second example, consider the longitudinal responses, pitch rate, and normal acceleration to elevator stick force, for two versions of an advanced fighter presented in Table 2.[2] The two versions differ only by the inclusion of a first-order prefilter in the forward control path, plus a high-frequency oscillatory mode around 40 rad/s. In this example, the prefilter mode (3.36 rad/s) lies within the frequency range of interest ($\omega_1 = .1$, $\omega_2 = 10.0$ rad/s). To investigate how adding the prefilter affects the ability to obtain a good approximation for the mid-frequency component $G_{\text{MID}}(s)$, Hankel singular values for $G_{\text{MID}}(s)$ are given for each system, along with the critical ratios h_i/h_{i+1}. The addition of the prefilter reduces h_2/h_3 from 6.1 in the first case to 2.62 in the second. Considering a third-order approximation for the aircraft with the prefilter would raise the ratio to $h_3/h_4 = 5.38$, which, it turns out, is closer to that for a second-order approximation of the aircraft without the prefilter. The implication is that without the prefilter, the effective order of G_{MID} is two, and with the prefilter, the effective order is three, and this is consistent with the results in Ref. 2.

A more concrete result involving the internally balanced algorithm was proved by Enns,[9] who showed that the frequency response error of the rth order model is bounded for all ω by

$$\bar{\sigma}\{Q_o[G(j\omega) - G_r(j\omega)]Q_i\} < 2Tr(\Sigma_{n-r}) \tag{42}$$

where $\bar{\sigma}(\cdot)$ is the maximum singular value of $(\cdot)$, and $\Sigma_{n-r} = \text{diag}(h_j)$ where $j = r+1, n$. The bound is defined by the truncated Hankel singular values of the scaled system $Q_o G(s) Q_i$, which like $G(s)$ are invariant to state transformation. [The scaling matrices Q_o and Q_i will be discussed later, along with an interpretation of this bound in the context of the complete procedure. For now, note that if $h_r > h_{r+1}$, the subsystem approximation is unique, stable,[12] and satisfies Eq. (42).]

A State-Space Equivalent-Systems Procedure

The procedure proposed[13] is a hybrid of the two model reduction ideas presented previously, the modal decomposition

Table 2 Example HOS aircraft

Advanced fighter without prefilter	$(h_i)^{a}$	(h_i/h_{i+1})
$\frac{q}{\delta} = \frac{5.26(0)(0.0103)(0.773)(0.5)(1.887)(13.986)}{[0.016,0.082][0.61,2.78](0.418)(1.34)[0.97,17.04]} \left(\frac{1}{s}\right)$	.6005	1.66
	.3617	6.15
$\frac{n_{z_{cr}}}{\delta} = \frac{1.34(0)(0.00066)(49.99)(0.5)(1.887)(13.986)}{\Delta} \left(\frac{g}{rad}\right)$	.0588	7.82
	.0060	—
Advanced fighter with prefilter		
$\frac{q}{F} = \frac{141.1(39.815)(0)(0.0103)(0.773)(0.5)(1.887)(13.986)}{(3.366)[0.46,39.75][0.016,0.82][0.61,2.78](0.418)(1.34)][0.97,17.04]} \left(\frac{rad}{lb\ s}\right)$	.1203	1.80
	.06692	2.62
$\frac{n_{z_{cr}}}{F} = \frac{35.946(0)(0.00066)(49.59)(0.5)(1.887)(13.985)(39.8)}{\Delta} \left(\frac{g}{lb}\right)$	.02559	5.37
	.00476	14.56
	.000327	—

[a]Hankel singular values of G_{MID}: $(d_1 = 0.1,\ d_2 = 10)$, $Q_o = \mathrm{diag}[10,1]$.

Table 3 Modal decomposition

Let radii d_1 and d_2 (rather than the ω_1 and ω_2 shown in Fig. 5) define concentric circles in the complex plane. Also,

$$G(s) = C(sI - A)^{-1}B$$

Modally decompose A, to obtain Λ and M, where Λ is the block diagonal, real Jordan form and M is the corresponding (real) modal matrix.

The columns of M are now ordered according to the natural frequency $\omega_{n_i} = (\sigma_i^2 + \omega_i^2)^{\frac{1}{2}}$ of the corresponding modes. This defines

Regions	Column groupings
LO: $0 \le \omega_{n_i} < d_1$	M_{LO}
MID: $d_1 \le \omega_{n_i} < d_2$	M_{MID}
HI: $d_2 \le \omega_{n_i}$	M_{HI}

Let

$$V \triangleq [M_{\mathrm{LO}} \quad M_{\mathrm{MID}} \quad M_{\mathrm{HI}}]$$

$$V^{-T} \triangleq [Z_{\mathrm{LO}} \quad Z_{\mathrm{MID}} \quad Z_{\mathrm{HI}}]$$

The subsystems $G_i(s) = C_i(sI = A_i)^{-1}B_i$, where i = LO, MID, HI are now

$$A_i = Z_i^T A M_i, \quad B_i = Z_i^T B, \quad C_i = CM_i$$

and the internally balanced approach. First, the system $G(s)$ is modally decomposed into three subsystems $G_{\mathrm{LO}}(s)$, $G_{\mathrm{MID}}(s)$, and $G_{\mathrm{HI}}(s)$, denoting the low-, mid-, and high-frequency components of the system, as described in Table 3. The input/output scaling (i.e., diagonal matrices Q_i and Q_o) is then chosen to weight the responses of interest, and G_{MID} and G_{HI} are reduced using the internally balanced technique in Table 1. A first-order approximation is sought for G_{HI}. whereas an approximation of the desired "classical" order is sought for G_{MID}. The complete reduced-order approximation is the sum of these two subsystem approximations.

The solution is uniquely determined by relatively few parameters: 1) r_{MID}, r_{HI}; the order of G_{MID_r}, G_{HI_r}. 2) d_1,d_2; the radii of the concentric circles which define G_{MID} and G_{HI}. 3) Q_i,Q_o; the i/o scaling. The parameters r_{MID} and r_{HI} define the desired order pertaining to the approximations for $G_{\mathrm{MID}}(s)$ and $G_{\mathrm{HI}}(s)$, respectively. It is desirable that r_{MID} correspond to the "classical" order of the responses being modeled. For longitudinal responses, $r_{\mathrm{MID}} = 2$ may correspond to a short-period approximation. For lateral/directional responses, $r_{\mathrm{MID}} = 3$ may correspond to an approximation for the aircraft's dutch/roll and roll subsidence modes.

The parameters d_1 and d_2 divide the complex s-plane into three regions, and define the subsystems $G_{\mathrm{LO}}(s)$, $G_{\mathrm{MID}}(s)$, and $G_{\mathrm{HI}}(s)$. For longitudinal dynamics, d_1 must be such that $G_{\mathrm{LO}}(s)$ contains any phugoid mode, for example. (Note that this may be larger than ω_1.) For lateral/directional dynamics, d_1 must be such that $G_{\mathrm{LO}}(s)$ contains the spiral mode. The choice of d_2 determines the modes in $G_{\mathrm{MID}}(s)$ and $G_{\mathrm{HI}}(s)$. Sensitivity of LOS parameters with d_2 is a question of whether a specific pole/residue partial fraction significantly changes the reduced-order model if it is included in $G_{\mathrm{HI}}(s)$, as opposed to $G_{\mathrm{MID}}(s)$. If the modal participation of the mode in both subsystems is insignificant, then varying d_2 about this mode produces only minor changes.

The measure of how well the LOS model approximates the original HOS is reflected in the model's frequency response error bound. The norm described in Eq. (12) obeys the triangle inequality; that is, since

$$[G(j\omega) - G_r(j\omega)] = [G_{\mathrm{MID}}(j\omega) - G_{\mathrm{MID}_r}(j\omega)] + [G_{\mathrm{HI}}(j\omega) - G_{\mathrm{HI}_r}(j\omega)] + G_{\mathrm{LO}}(j\omega) \tag{43}$$

then

$$\bar{\sigma}[E(j\omega)] \le \bar{\sigma}[E_{\mathrm{MID}}(j\omega)] + \bar{\sigma}[E_{\mathrm{HI}}(j\omega)] + \bar{\sigma}[G_{\mathrm{LO}}(j\omega)] \tag{44}$$

If $G_{\mathrm{LO}}(s)$ contribution to the frequency response in the region $\omega_1 < \omega < \omega_2$ is negligible, the frequency response error of the reduced-order model described above is bounded, similar to Eq. (42), by

$$\sup_{\omega_1 < \omega < \omega_2} \bar{\sigma}[Q_o E(j\omega) Q_i] \le 2[Tr(\Sigma_{\mathrm{MID}_{n-r}}) + Tr(\Sigma_{\mathrm{HI}_{n-r}})] \tag{45}$$

Furthermore, it can be shown[13] that if $Q_i = \mathrm{diag}(q_{i_j})$ and $Q_o = \mathrm{diag}(q_{o_j})$,

$$|E_{ij}(j\omega)| \le B_{\infty_{ij}} \quad (\omega_1 < \omega < \omega_2) \tag{46}$$

where

$$B_{\infty_{ij}} = \frac{2}{q_{o_i} q_{i_j}} [Tr(\Sigma_{\mathrm{MID}_{n-r}}) + Tr(\Sigma_{\mathrm{HI}_{n-r}})] \tag{47}$$

The relationship between Eq. (46) and the resulting Bode error has been discussed previously [i.e., in Eq. (14) let $\alpha_{ij} = B_{\infty_{ij}}/M_{\mathrm{II}}$]. In many cases, $|G_{ij}(j\omega)|$ is roughly constant over the frequency range of interest, so one α_{ij} can then be associated with a large portion of the frequency response.

Moreover, if the scaling (Q_i and Q_o) has been chosen such that elements of the matrix $G(j\omega)$ are weighted equally (their scaled magnitudes roughly equal), the same Bode error bound will be applicable for each element of the frequency response.

The need for scaling arises because the internally balanced approach is more sensitive to those responses with higher magnitudes. Different magnitudes can simply arise from different units in the input or output channels, or different force gradients in manipulators (aileron stick, rudder pedal). Consequently, the relative importance of responses should not be gauged by relative magnitude alone. Therefore, to obtain a uniform match between the truly dominant responses of the system, scaling must be included. Q_i and Q_o are square diagonal matrices containing the non-negative scaling factors q_{i_j} and q_{o_j}. By using scaling, one can also bias the resulting low-order system to better approximate a certain response over another, if desired. However, the present approach uses scaling to adjust the magnitudes of $G_{ij}(j\omega)$ to be approximately equal over the interval (ω_1,ω_2).

Lower-order approximations for the advanced fighter with and without prefilter in Table 2 are given in Table 4, and the corresponding frequency responses in Figs. 6–9. Also given are the equivalent system approximations.[2] There is good correspondence between results from the current methodology and those obtained via this new noniterative approach. In Ref. 2, Bischoff noted, for example, that an equivalent system of third order, rather than second, was required to obtain an adequate match for the case with the prefilter. This fact was indicated by the Hankel singular values discussed along with Table 2. (Examples involving lateral-directional dynamics and a 2×2 transfer matrix are given in Refs. 10 and 13.)

In this example, the a priori error bounds [Eq. (46)] corresponding to the aircraft without prefilter dynamics were determined to be

$$|E_q(j\omega)| \le 0.01368$$

$$|E_{n_z}(j\omega)| \le 0.1368$$

for all ω. The factor of 10 difference in bounds results from scaling chosen to make the order of magnitude of q and n_z roughly the same. These bounds tended to be conservative, especially for $E_q(j\omega)$. The actual bounds over the frequency range of interest were calculated to be

$$\left.\begin{aligned} |E_q(j\omega)| &\le 0.00486 \\ |E_{n_z}(j\omega)| &\le 0.118 \end{aligned}\right. \quad 0.1 < \omega < 10$$

using the resulting model.

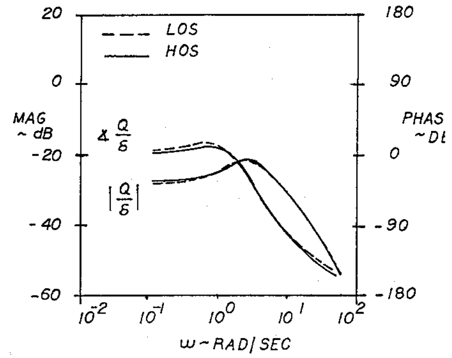

Fig. 6 Advanced fighter without prefilter: pitch rate to stick deflection frequency response.

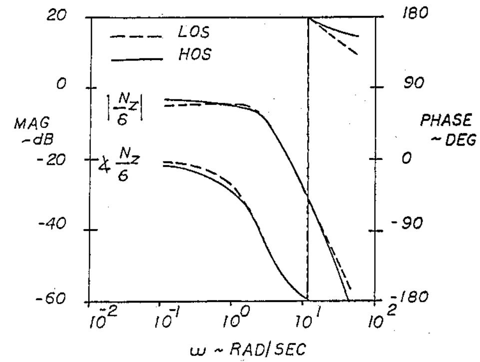

Fig. 7 Advanced fighter without prefilter: normal acceleration to stick deflection frequency response.

Table 4 Low order approximations

Advanced fighter without prefilter		
LOS with pole:		
$\dfrac{q}{\delta}=\dfrac{0.00251(0.882)(2376)}{(23.14)[0.657,2.45]}$		
$\dfrac{n_{z_{cr}}}{\delta}=\dfrac{-0.4292(31.55)(-51.53)}{\Delta}$		
With equivalent delay:		Equivalent System (with T_{θ_2} fixed)
$\dfrac{q}{\delta}=\dfrac{0.258(0.882)e^{-0.0428s}}{[0.657,2.45]}$	...	$\dfrac{0.277(0.773)e^{-0.0525s}}{[0.76.2.36]}$
$\dfrac{n_{z_{cr}}}{\delta}=\dfrac{0.0956(31.55)e^{-0.0626s}}{\Delta}$		
Advanced fighter with prefilter		
LOS third order, with delay:		Equivalent System (with T_{θ_2} fixed)
$\dfrac{q}{F}=\dfrac{0.154(0.761)e^{0.s}}{(1.483)[0.604,3.164]}$	...	$\dfrac{0.157(0.773)e^{-0.036s}}{(1.41)[0.62,3.28]}$
$\dfrac{n_{z_{cr}}}{F}=\dfrac{-0.0032(-20.4)(29.6)e^{0.s}}{\Delta}$		

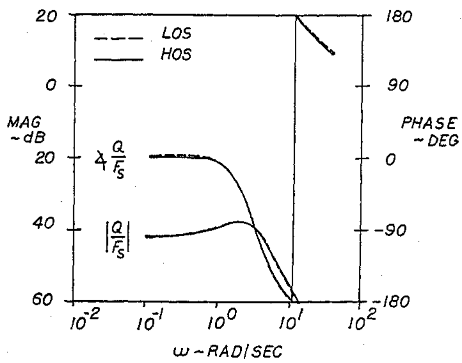

Fig. 8 Advanced fighter with prefilter: pitch rate to stick force frequency response.

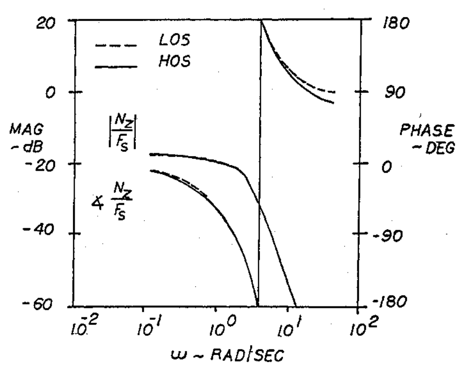

Fig. 9 Advanced fighter with prefilter: normal acceleration to stick force frequency response.

The a priori error bounds for the third-order match of the aircraft with prefilter dynamics are tighter

$$|E_q(j\omega)| \le 0.00106$$

$$|E_{n_z}(j\omega)| \le 0.0106$$

which is reflected in the better matches shown in Figs. 8 and 9. These bounds are tighter in the sense that α is smaller, and it should also be noted that the frequency-response magnitudes have changed significantly as well.

Summary and Conclusion

The equivalent-systems problem involves two key issues: the characteristics of the vehicle that are important to the pilot in the evaluation of some task, and the existence of a lower-order model of some specified form that preserves these characteristics. The modeling objective of the current equivalent-systems approach was re-examined in the context of a closed-loop system analysis of the pilot/vehicle system. An example was provided to underscore the critical significance of the vehicle's frequency response over the pilot-plus-vehicle region of crossover. Regarding existence, it was noted that not all high-order systems (aircraft) can be accurately modeled by equivalent systems of a given form and order. For a given frequency-response error, each system has an effective order. A set of parameters, the Hankel singular values, that gage this order were examined. Related questions concerning the goodness of fit required, as well as the relative importance of responses relevant to the task being evaluated, still remain topics for further research.

A procedure was offered that used the internally balanced approach to provide low-order approximations. This procedure conceptually provides the same type of reduction as the current methodology. In contrast, however, the procedure presented requires no initial guess and no iterative search. Furthermore, the maximum magnitude of the frequency-domain model error is bounded, rather than minimizing some average error, and this bound is available before any model is obtained. Such a state-space procedure that is well-founded in the frequency domain can be used to obtain error-bounded low-order approximations, thus opening the equivalent-systems concept to more complex multiloop tasks.

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References

[1]Hodgkinson, J., LaManna, W. J., and Heyde, J. L., "Handling Qualities of Aircraft with Stability and Control Augmentation Systems—A Fundamental Approach," *Aeronautical Journal*, Feb. 1976, pp. 75–81.

[2]Bischoff, D. E., "Development of Longitudinal Equivalent System Models for Selected U.S. Navy Tactical Aircraft," Rept. NADC-81069-60, Aug. 1981.

[3]Harper, R. P., "Handling Qualities of Flight Vehicles," Arvin/Calspan Advanced Technology Center, Buffalo, NY, 1984.

[4]McRuer, D., Graham, D., Krendal, E., and Kreisner, W., "Human Pilot Dynamics in Compensatory Systems—Theory, Models, and Experiments with Controlled Element and Forcing Function Variations," Flight Dynamics Laboratory, Wright-Patterson Air Force Base, OH, AFFDL-TR-65-15, July 1965.

[5]Neal, T. P. and Smith, R. E., "An In-flight Investigation to Develop System Design Criteria for Fighter Airplanes, Vol. 1, Flight Dynamics Laboratory, Wright-Patterson Air Force Base, OH, AFFDL-TR-70-74, Dec. 1970.

[6]Bacon, B. J. and Schmidt, D. K., "Multivariable Frequency Weighted Order Reduction," *Journal of Guidance, Control, and Dynamics*, (to be published).

[7]Doyle, J. C., and Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Transactions on Automatic Control*, Vol. AC-26, Feb. 1981.

[8]Moore, B. C., "Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction," *IEEE Transactions on Automatic Control*, Vol. AC-26, Feb. 1981.

[9]Enns, D. F., "Model Reduction for Control System Design," Ph.D. Dissertation, Dept. of Aeronautics and Astronautics, Stanford Univ., Stanford, CA, March 1984.

[10]Bacon, B. J., "Model Reduction Appropriate of Linear Closed-loop Systems Analysis," Ph.D. Dissertation, School of Aeronautics and Astronautics, Purdue Univ., West Lafayette, IN, 1988.

[11]Kailath, T., *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.

[12]Pernebo, L. and Silverman, L. M., "Model Reduction in Balanced State-Space Representation," *IEEE Transactions on Automatic Control*, AC-27, 1982, pp. 382–387.

[13]Bacon, B. J. and Schmidt, D. K., "Mathematics of Equivalent Models for Aircraft Dynamics," School of Aeronautics and Astronautics, Purdue Univ., West Lafayette, IN, Final Rept., Naval Air Development Center Contract N62269-83-C-0220, July 1987.